Segment of a rainbow showing supernumerary bows. Note darkness of sky outside the bow.

Rainbow on a spreadsheet

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This past spring a colleague of mine, a chemist, had undertaken to explore a subject trendy among college math teachers--calculus reform. One idea for calculus reform is to provide more relevant problems; and, among the several sample problems my colleague was given was one of finding the angular radius of the rainbow. He asked me, "How would you start such a problem?"

Thinking that I completely understood the problem of the rainbow, I gave a flippant answer, such as, "Oh, it's a problem in finding a stationary phase or path length extremum." But, as I thought about the problem I wondered how I could make a simple explanation of the rainbow as a problem in classical scattering. Scattering is a subject familiar to both physicists and geophysicists, but I have never seen it used for a simple explanation of the rainbow.

Explanations of the rainbow

There are several theories of the rainbow, each more sophisticated than the prior one, and explaining more of the rainbow's myriad features. My analysis takes a big step back to the theory that DesCartes published in 1637[2]. Despite this being the most elementary of rainbow theories, it has the distinction of explaining many of the rainbow's salient features without using physics and mathematics beyond the reach of most college undergraduates. It is similar to the common explanation found in textbooks and popular science literature. However, I now see that this theory, as it is ordinarily presented in textbooks, contains a flaw in its logic.

Descartes' theory uses geometric optics. Geometric optics is an approximate theory in which light travels along paths called rays. In uniform materials these rays are straight lines except where they intercept a surface and are reflected or refracted. This theory cannot explain phenomena like diffraction, that depend for explanation entirely on the wavelike nature of light. Yet, despite its simplicity, geometric optics can explain the operation of lenses and mirrors; and it explains perfectly well phenomena that depend on interference. All that a person needs to use geometric optics is geometry, the law of reflection, and Snel's law.

The common geometric explanation of the rainbow, essentially, is to show that a bundle of light rays penetrating a raindrop, and suffering one refraction at incidence, one internal reflection, and a second refraction at exit, have an extreme deviation at an angle equal to the radius of the primary rainbow. This is not how DesCartes actually approached the problem, and, in fact, it presents flawed logic in the following sense.

We see a rainbow because scattered light happens to be especially intense in the direction of the rainbow. Finding a ray of minimum deviation does not necessarily provide an explanation of why skylight is bright along that ray.

DesCartes himself began his investigation by tracing 10 parallel rays through a droplet. He placed these rays parallel to the axis of the raindrop, and spaced them by 10% of the raindrop radius. He found that the 8th and 9th rays left the droplet at nearly the same angle of 41°, which is approximately the radius of the rainbow. However, DesCartes realized that this calculation alone does not explain why the sky is bright in the direction of the rainbow. Consequently he did a detailed study by tracing more tightly spaced rays on the interval between his 8th and 10th original rays. DesCartes' own words express his findings better than any subsequent account.

"I took my pen and made an accurate calculation of the paths of the rays which fall on the different points of a globe of water to determine at what angles, after two refractions and one or two reflections they will come to the eye, and then I found that after one reflection and two refractions there are many more rays which can be seen at an angle of from forty-one to forty-two degrees than at any smaller angle; and that there are none which can be seen at a larger angle."

What DesCartes says, in effect, is that the droplet diverts more energy into the direction of the rainbow than into any other direction, which is the correct answer to the question of why we see a rainbow. In order to show why a rainbow appears for the particular case of two refractions and one reflection, we must show how light energy becomes concentrated along a particular direction; and, one way of doing this is to treat the rainbow like a classical problem in scattering.

Scattering problems

The prototypical scattering problem is Ernest Rutherford's analysis of how alpha particles deflect from a dense, charged sphere, which lead to the discovery of the atomic nucleus in 1913. Physicists and geophysicists alike deal commonly with scattering problems. Examples include: finding the distribution of particles that leave the target of an particle accelerator, or finding the distribution of X-rays that scatter from the human body to produce an X-ray tomogram, or finding the distribution of sound energy that leaves a submarine lurking in the ocean depths. These appear as very different problems, but each involves a target that scatters some type of radiation.

Scattering problems come in two varieties. A forward problem is one in which we know everything about the incident radiation and the target, and we can calculate how the radiation will scatter. More commonly in real situations we have an inverse problem. In this we know all about our incident radiation and we can measure the scattered radiation. From these two pieces of information we infer what the thing causing the scattering must be like. In order to solve the inverse problem, we must be able to solve the forward problem for at least one model of the target.

How does the rainbow tally up as a problem in scattering? First, we know all that we need about the incident radiation, sunlight. For example, it comes from a relatively small source, and so, its rays are uniform across a small section of sky, and form a parallel bundle. If we wish to specify what sunlight is like more accurately we can treat the rays as making up a distribution in which green rays are more abundant than those of other colors. A more accurate description yet requires us to treat the rays as having a wavelength and the ability to interfere with one another, and so forth.

Second, we know just about all there is to know about the scattered sunlight, because we have two millennia of observations of rainbows. For instance, we know that the scattered radiation has a peak intensity in the direction of 138ø from the incident radiation. We know that there are supernumerary bows inside the primary, and that the sky is especially dark outside the primary bow, and so forth, and so on.

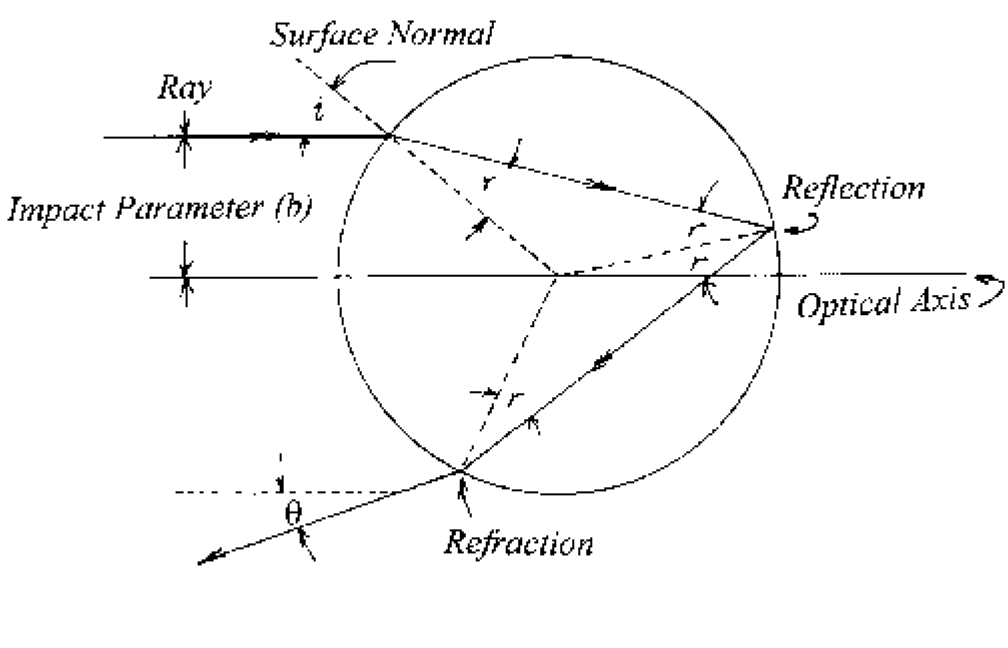
Third, we know how to solve the forward problem for the model of a spherical raindrop. We simply apply Snel's law to the refraction of an incident ray on the raindrop surface, then apply the ordinary law of reflection where the ray reaches the raindrop surface a second time, and finally apply Snel's law where the ray leaves the raindrop again.

How do we repeat DesCartes' calculation of the rainbow without resorting to his tedious method of tracing rays, and at the same time make it explicitly a scattering problem?

First, draw a figure that shows an arbitrary ray incident on a droplet and trace its path according to Snel's law of refraction at the surface of the droplet, and the law of reflection at the intermediate point. Second, write the function of scattering angle as a function of incident angle. Referring to Figure 2,below, this is $\theta = 2(i-r)+2(\pi-r)$. Finally, we use ordinary calculus to find the fraction of incident energy that scatters into any direction. Calculus enters only at this third step.

The spherical raindrop makes a good example of a forward scattering problem. Figure 2 shows a spherical droplet illuminated with a beam of light rays that propagate parallel to a diameter of the droplet. How do we specify any given ray in the beam? One obvious way is to use the distance between the axis of the droplet and the point at which the ray intercepts the surface. This is exactly what DesCartes did. In scattering problems we call the distance b the impact parameter. However, the ratio of b/R is also unique to the ray and could serve as suitable impact parameter. In fact b/R is the sine of angle of incidence.

Now we know that for each value of impact parameter (angle i in this case) the ray leaves the droplet along some other angle(θ). Energy that impacts the droplet scatters into space along new rays. A simple extension of this idea is that rays having a small range of impact parameter beginning at b and extending tob+db scatter into rays leaving the droplet beginning at angle é and extending over a small range to θ+dθ. If we multiply db by 2πb, we have a small annular area on the face of the droplet. Physicists call the ratio (2πb)db/dθ the differential scattering cross section of the droplet. This is a very learned term for something very simple. Differential cross section tells us the amount of energy that scatters along any particular angle é. So the essence of a forward scattering problem consists entirely of finding the derivative db/dθ.



A rainbow on a spreadsheet

Since the ratio db/dθ is a derivative, our analysis seems to require calculus. However, we can use a spreadsheet to solve the problem which is much more in the spirit of DesCartes' original method. The spreadsheet below has 7 columns of data for this problem. Each column corresponds to a quantity of interest. Each row corresponds to a single ray traced through a droplet that begins with a different angle of incidence. By changing the index of refraction used in column b, the spreadsheet will calculate scattering for a different color of light, or a different material.

Table 1 Scattering Calculations Spherical Raindrop

angle(i)	angle(r)	θ	B	db	dθ	abs(db/dθ)
0.00	0.00	0.00	0.00			
5.00	3.75	5.00	0.09	0.17	9.94	0.99
10.00	7.49	9.94	0.17	0.17	9.79	0.99
15.00	11.20	14.78	0.26	0.17	9.53	1.00
20.00	14.87	19.47	0.34	0.16	9.15	1.01
25.00	18.48	23.94	0.42	0.16	8.65	1.03
30.00	22.03	28.12	0.50	0.15	8.01	1.07
35.00	25.49	31.94	0.57	0.14	7.20	1.12
40.00	28.83	35.32	0.64	0.13	6.20	1.22
45.00	32.04	38.15	0.71	0.12	4.99	1.40
50.00	35.08	40.31	0.77	0.11	3.52	1.80
55.00	37.92	41.67	0.82	0.10	1.76	3.21
60.00	40.52	42.07	0.87	0.09	-0.32	15.21
65.00	42.84	41.34	0.91	0.07	-2.77	1.50
70.00	44.83	39.30	0.94	0.06	-5.59	0.60
75.00	46.44	35.75	0.97	0.05	-8.79	0.29
80.00	47.63	30.51	0.98	0.03	-12.31	0.14
85.00	48.36	23.44	1.00	0.01	-10.44	0.07
87.00	48.52	20.07	1.00	0.00	-7.05	0.03
89.00	48.60	16.39	1.00			

Detailed Scattering Calculations

angle(i)	angle(r)	θ	B	db	dθ	abs(db/dθ)
58.00	39.51	42.03	0.85			
58.50	39.76	42.06	0.85	0.01	0.04	13.00
59.00	40.02	42.07	0.86	0.86	0.87	0.87
59.50	40.27	42.08	0.86	0.01	0.00	124.28
60.00	40.52	42.07	0.87	0.01	-0.03	18.68
60.50	40.76	42.05	0.87	0.01	-0.05	9.88
61.00	41.01	42.02	0.87	0.01	-0.07	6.62
61.50	41.24	41.98	0.88	0.01	-0.10	4.92
62.00	41.48	41.93	0.88	0.01	-0.12	3.87
62.50	41.71	41.86	0.89			

On computer spreadsheets columns are usually designated by lower case letters, and I'll follow that convention here. Column a is the incident angle (i) of a ray. It is simply a number that the user can specify (90 degrees is a problem so we skip it).Column b is the angle of the refracted ray (r) computed from Snel's law. Column c is theta (θ) as computed from our relation involving angles (i) and (r). Column d contains the impact parameter, b, which I compute from the relation b=R*sin(i), assuming R=1. DesCartes would have used R=10,000, but the spreadsheet has no aversion to decimals. Column e is the central difference of b and is a good approximation to db. Likewise, column f is the approximate differential of θ, dθ. The last column (g) is the ratio of db/dθ multiplied by a factor that makes the product approach 1 as (i) approaches zero. In this form column g is an estimate of how much the sky is brightened in the direction (θ) compared to the antisolar point. Note that I use the absolute value of db/dθ in this column, because whether the rays are inverted or upright has no bearing on their intensity. Figure 3, shown below, illustrates how most rays entering a spherical droplet, exit in a very consistent direction. This spreadsheet is, in fact, doing calculus. We just don't mention it. Table 2 indicates the calculations at each spreadsheet cell.

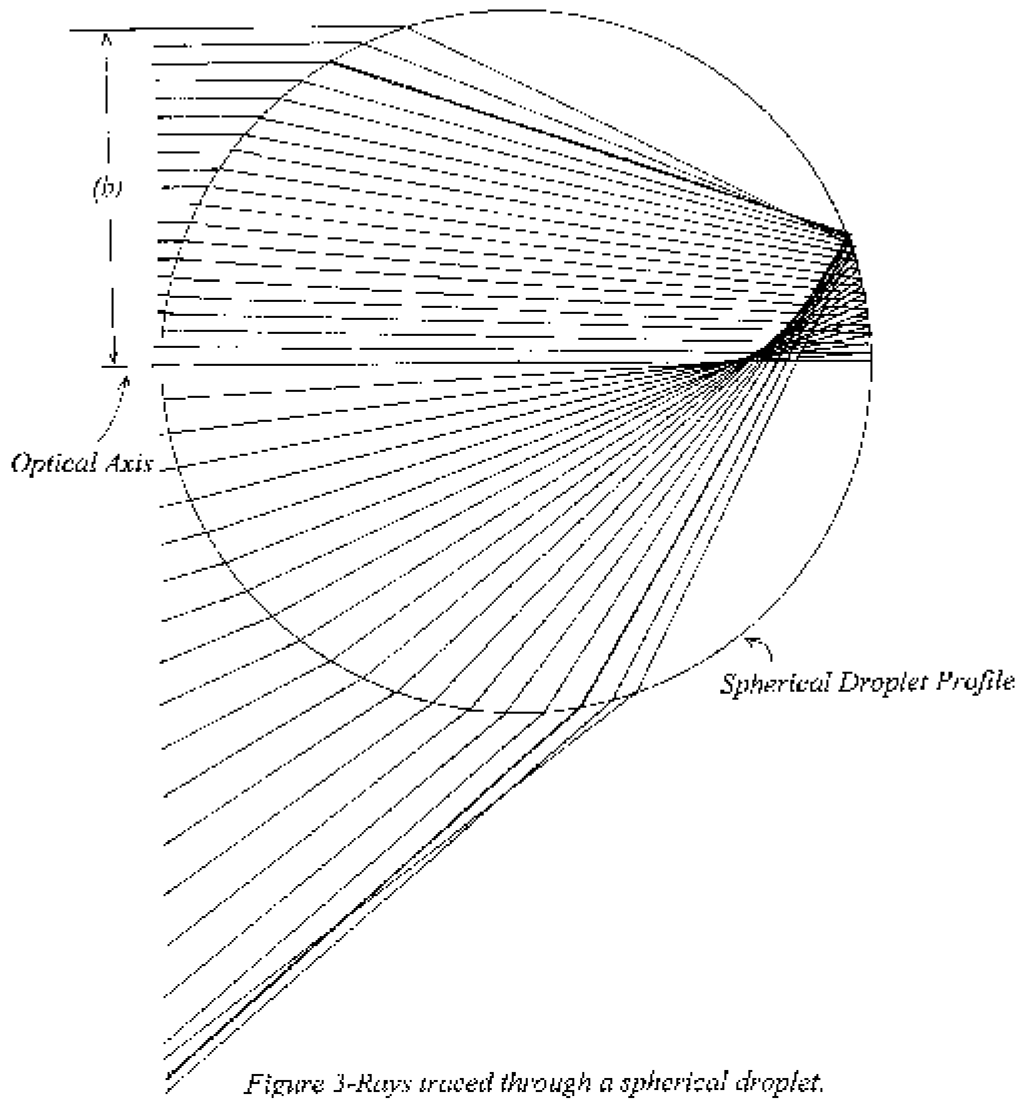


Figure 3-Rays traced through a spherical droplet.

Table 2. Cell functions

Col.	Function (J=current row)	Note
a	Numeric angle of incidence	
b	@asin(@sin(@rad(aJ))/1.333)	Snel's Law
c	4*bJ-2*aJ	angle of deviation (θ)
d	@sin(@rad(aJ))	b (assuming radius=1)
e	+c(J+1)-c(J-1)	approximation to db
f	+d(J+1)-d(J-1)	approximation to dé
g	factor*eJ/fJ	factor to make result->1

Note: 1.333 is an assumed index of refraction. Replace with value desired per color or material.

The simple analysis of the angular radius of the rainbow is finished at this point. I have shown why a rainbow appears at an angular radius 42ø from the antisolar point. The answer is, because the raindrop has a large scattering cross section for these particular rays. Most textbook authors generally just assume that the least deviated ray is also very bright. Here, I have shown it to be so. The ray of minimum deviation happens to occur where the scattering cross section is greatest. Yet, looking for the minimum deviated ray only seems to find the correct rainbow radius in a search for the wrong thing.

Notes:

1 - Carl Boyer also mentions this same problem, but only in passing. I have no idea if his mention of it spurred others into an examination of the problem. Carl Boyer, The Rainbow, Princeton U. Press, 1957.

2 - Rene' Descartes began his investigation into the cause of the rainbow in the spring of 1629, but did not publish his theory until 1637.

3 - Rays take curved paths in material that has a gradient in its index of refraction. This is how mirages form. The Law of Reflection states that the angle of incidence equals the angle of reflection. These angles are measured between the ray and a perpendicular to the surface at the point of reflection. Snel's Law states that the product of index of refraction and sine of the angle of incidence remains constant for a ray throughout its propagation. In seismology, where the rays are acoustic and their source is earthquakes, the product of index of refraction and sine of angle of incidence is a constant called the ray parameter.

4 - The reader may find this sort of explanation in Robert W. Wood, Physical Optics, Dover Publ., 1967, as well as in W. J. Humphreys, Physics of the Air, Dover Publ., 1964. Both of these authors, however, also present a complete explanation of the rainbow using the wave theory of light. However, this idea of finding a ray of extreme deviation is common among textbook explanations. Undoubtedly it is what the authors of calculus reform have in mind.

In fact, in doing research for this paper, I found textbook explanations that run the range of nearly correct to absolutely wrong. For example, Joseph M. Moran and Michael D. Morgan, Meteorology, 2nd Ed. MacMillan Publ., New York, 1986, refer only to the dispersion of light, as if the separation of colors alone is sufficient explanation. F.K.Lutgers and E.J.Tarbuck (The Atmosphere, 4th ed., Prentice-Hall, 1989) use this same argument initially, but eventually also mention that light rays are crowded together in the direction of the rainbow. However, their diagram suggests a crowding of rays at the exit point on the droplet, not a crowding in the direction of exit. C. Donald Ahrens (Essentials of Meteorology, West Publ., 1993), which probably is the most popular meteorology text in college courses, asserts that the rainbow results from rays near and beyond total internal reflection in a droplet. Yet, there is no total internal reflection within a spherical raindrop. Total internal reflection has nothing to do with the rainbow.

5 - Some historical and scientific authors have misrepresented DesCartes' account to suggest that he calculated the paths of 10,000 rays. However, what he actually did was to set the radius of the droplet at 10,000 units to avoid decimals.
6 - This account is from Boyer's book p. 211. The italics are from Boyer's account, but for some reason he does not explain the significance of the passage which he italicized.
7 - Rutherford actually calculated this scattering problem in1911. The actual discovery of the nucleus is usually placed in1913 after Geiger and Marsden made more complete experiments. Any 3rd year college physics text in mechanics contains an explanation of this problem.
8 - By ordinary calculus I mean not calculus of variations.