1 Steady-state tornado vortex models

2 The Rankine combined vortex

by: Kevin T. Kilty
June, 2005

The Rankine combined vortex is a simple model possessing only azimuthal velocity. Unfortunately, without non-azimuthal velocity components there is no mechanism that can produce a Rankine vortex in the real atmosphere. Yet, it does manage a surprisingly accurate description of air flow around a tornado just the same.

The Rankine vortex is often called a Rankine combined vortex for the reason that it has two separate flow fields. The interior flow field (core) involves only aazimuthal velocity which increases linearly with radius from zero along the central axis to a maximum value at a radius (R). Thus this region rotates like a solid body even though it is fluid. The outer flow (tail) is also purely azimuthal with maximum velocity at radius R. The velocity declines inversely with radius from this point outward. Such a flow is called a *potential flow* because there is a scalar velocity potential function for it. The mathematical description of the combined vortex is:

$$
\begin{aligned}
V_\theta(r) &= \frac{V_0 r}{R} \ (r < R); \text{and} \\
V_\theta(r) &= \frac{V_0 R}{r} \ (r > R);
\end{aligned}
\tag{1}
$$

where, V_0 = vortex strength (azimuthal velocity at core boundary[1]), r= radial coordinate, and R = radius of the vortex core; or, alternatively one might use the stream functions $\Psi_{core} = \frac{V_0 r^2}{2R}$ and $\Psi_{tail} = V_0 R Log(r)$ where $V_\theta = \partial\Psi/\partial r$. Tail flow being derivable from a scalar potential (Φ) means, mathematically, that $\vec{V} = -\nabla\Phi$ or $V_\theta = -\frac{\partial\Phi}{r\partial\theta}$. This, used with the definition of vorticity, $\vec{\omega} = \nabla \times \vec{V}$ combined with the vector identity $\nabla \times \nabla\vec{A} = 0$

[1]Rather than use V_0 people often use circulation, Γ, in the combination $\frac{\Gamma}{2\pi r_e}$ where r_e is some effective radius of circulation

indicates that the tail of the Rankine combined vortex has no vorticity! All vorticity belongs to the core flow. The central core of the vortex has constant vorticity throughout.

Even though there is no means of producing such a vortex in the atmosphere, the Rankine vortex describes air flow observed in movies of the Dallas (2 Apr 1957) Tornado very well at the 1000 foot level.[2]

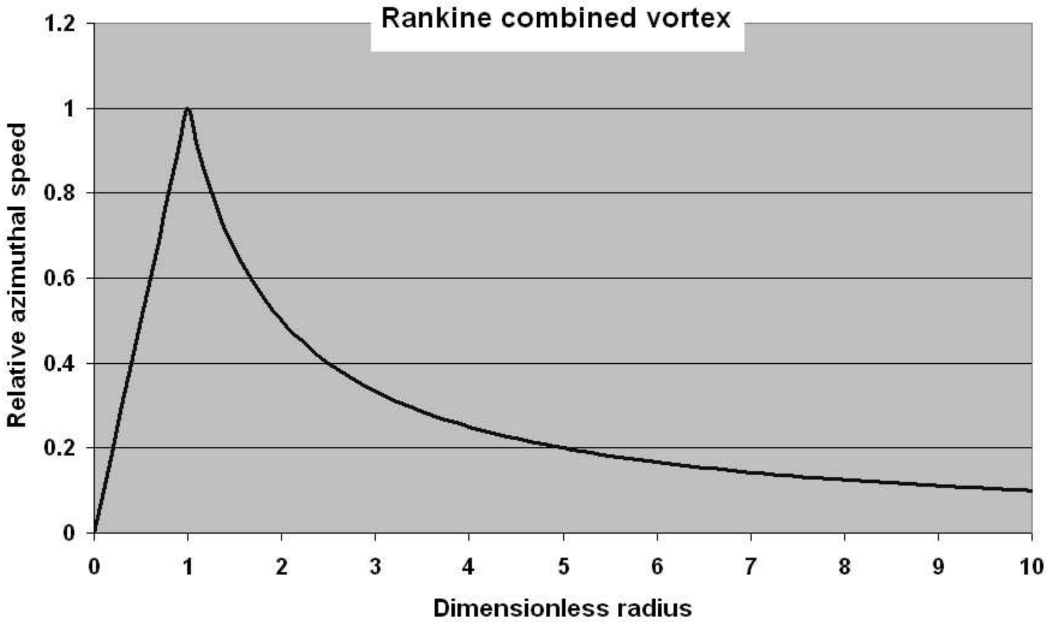

Figure 1: Form of azimuthal velocity in a combined Rankine vortex.

A person might reasonably wonder about how to attach a numerical value to the single parameter (V_0) in a description of the combined Rankine vortex. One way would be to measure the maximum azimuthal winds. Or, a person could estimate the value of V_0 from the central pressure deficit in a tornado. But since either approach depends on quantities that are difficult to mea-

[2]At lower levels in this tornado there are important departures from the combined Rankine description suggestive of Starr's idea of radially inward transport of momentum.

sure, to say the least, an indirect estimate comes from equating the convective atmosphere potential energy (CAPE), a thermodynamic quantity available on atmospheric soundings, to maximum kinetic energy per unit mass of air, which is to say $\frac{1}{2}v^2$.[3] As an example, consider the devastating Spencer, South Dakota tornado of May 30, 1998. A sounding made at Valley, Nebraska a few hours prior to the tornado provided a CAPE of about 3400 J/Kg. This suggests air could achieve a maximum speed of something like 82 m/s by completely converting CAPE into motion. Damage and doppler radar measurements of the Spencer tornado suggested $F4$ intensity, and wind speeds of 110 m/s. Thus the estimate derived from CAPE is very reasonable, and there are good arguments, some based on Starr's idea, about why wind speeds will exceed the thermodynamic limit near the ground surface which I will discuss in the separate page about stability.[4]

2.1 Rankine vortex stability

Lord Rayleigh in 1916 and other investigators since this time have shown that an arbitrary azimuthal velocity in an inviscid fluid ($\nu = 0$ or $\mu = 0$) , $V(r)$, having no dependence on θ is stable or unstable against axi-symmetric (no θ dependence) distubances accordingly as the following conditions hold true. Let a function, $\Omega(r)$, equal $\frac{V(r)}{r}$, then

$$\begin{aligned} \text{If } \frac{d^2}{dr^2}(r^2\Omega)^2 > 0 \text{ the flow is stable} \\ \text{If } \frac{d^2}{dr^2}(r^2\Omega)^2 < 0 \text{ the flow is unstable} \end{aligned} \tag{2}$$

The case $\frac{d^2}{dr^2}(r^2\Omega)^2 = 0$ would appear to be neutral stability. The effect of viscosity (molecular or eddy) is to stabilize the motion. With a bit of

[3]While CAPE actually measures the energy available to lift air parcels, a person might also think of CAPE as being the energy capable of supporting a pressure deficit within the core of a mesocyclone or a tornado.

[4]The forward motion of the tornado itself will add to the maximum windspeed on one flank of the tornado. In the case of the Spencer storm this amounted to an additional 15 m/s

viscosity taken into account the condition for stability ($\frac{d^2}{dr^2}(r^2\Omega)^2 > 0$) still holds, but the condition $\frac{d^2}{dr^2}(r^2\Omega)^2 < 0$ no longer guarantees instability.

By substituting the inner core flow for V(r) in the stability criterion, a person can easily show that the core of the Rankine vortex is stable (see the next page in this series on stability of radial disturbances). The tail flow is neutrally stable, however a little viscosity should stabilize it. However, keep in mind that total viscosity involves the sum of molecular and eddy viscosity, and in some cases eddy viscosity might become negative, and a negative viscosity would probably de-stabilize the flow.[5]

3 The Burgers-Rott vortex

This is an exact solution to the Navier-Stokes Equation, and assumes the mathematical form

$$\begin{gathered} U(r) = -ar \\ V(r) = \frac{\Gamma}{2\pi r}(1 - e^{-\frac{ar^2}{2\nu}}) \\ W(z) = 2az \\ P(r,z) = P_0 + \rho\int_0^r \frac{\nu^2}{r}dr - \frac{\rho a^2}{2}(r^2 + 4z^2) \text{ where;} \end{gathered} \tag{3}$$

U,V, and W are the r, θ, and z components of velocity, respectively. Γ is circulation strength of the vortex and a is the strength of suction. $P(r,z)$ is the distribution of atmospheric pressure. This vortex has a central axis like the Rankine vortex around which there is an azimuthal flow. However, unlike the Rankine vortex a Burgers-Rott vortex has radial and vertical flow as well. Air spirals in toward the axis and then flows upward. Unlike the Rankine vortex there is partially a mechanism for making a Burgers-Rott vortex in the atmosphere, as it results from suction at great height above a

[5] I have no proof of such, but I speculate that a bit of negative eddy viscosity will alter the stability criterion so that the condition $\frac{d^2}{dr^2}(r^2\Omega)^2 > 0$ will no longer guarantee stability, but that $\frac{d^2}{dr^2}(r^2\Omega)^2 < 0$ will guarantee instability.

plane surface. In the atmosphere a thunderstorm driven by intense convection could provide suction to draw air toward a point on the ground below and draw the air upward into the cloud. Therefore at first glance a Burgers-Rott vortex could approximate the air flow beneath the central region of a large thunderhead. Coupling between the azimuthal and corner flows involves the parameter a, which describes the strength of the suction of the overlying thunderhead. As the suction becomes greater the azimuthal flow more nearly approximates the potential flow of the combined Rankine vortex. In fact, the azimuthal velocity distribution approximates the combined Rankine vortex quite well, and even smooths out the troublesome cusp in velocity at the interface between the core and tail flows.(see figure 1). As a model of a real tornado the Burgers-Rott vortex suffers from many deficiencies. One is that it is too symmetric. Notice that there is no value of r that localizes the vortex. The vertical velocity is only a function of z which means the vertical velocity is not confined to any region at the surface, but is the same everywhere. There is no local thunderstorm in other words. Also, the strength of circulation, Γ, appearing as it does only in the azimuthal equation, is arbitrary, and uncoupled from the remaining flow in the model. Another deficiency is that the vertical velocity increases linearly with height without bound, which in effect places the source of suction at a very great height (infinity in fact); whereas buoyancy in a convecting atmosphere is distributed over a height range. Therefore suction in the Burgers-Rott sense would only apply to the lowest portions of a tornado, where the height to the LCL[6] seems very high and the areal extent of the thunderstorm seems extremely large in comparison. It would be especially inappropriate to apply to a mesocyclone or a rotating thunderstorm. Finally, the axial pressure gradient is $\partial P/\partial z = -4\rho a^2 z$, and increases vertically without bound. Obviously the pressure gradient must reverse somewhere above–where the winds eventually diverge.

Let us briefly examine some characteristics of the Burgers-Rott vortex. First, the azimuthal wind component reaches a maximum value where

$$\frac{1}{2\pi}(\frac{a}{\nu}e^{-ar^2/2\nu} - \frac{1-e^{-ar^2/2\nu}}{r^2}) = 0 \tag{4}$$

Or, near $r(a/2\nu)^{\frac{1}{2}} = 1.12$. The accompanying graph shows the form of the

[6]The lifting condesation level (LVL) acts as the locus of the suction in this case.

azimuthal component[7], and it is plain to see that the form is extremely similar to the piecewise continuous description of the combined Rankine vortex. Thus the peak azimuthal velocity defines the boundary of the core, and the tail region contains momentum but very little vorticity. To get an idea of what this implies for a value for a, assume a vortex with a radius of 50 m and an eddy viscosity of 5 m^2/s; a then is about 0.004.[8] If the peak azimuthal velocity is about 100 m/s, the circulation Γ is then about 1.7×10^4 m^2/s.

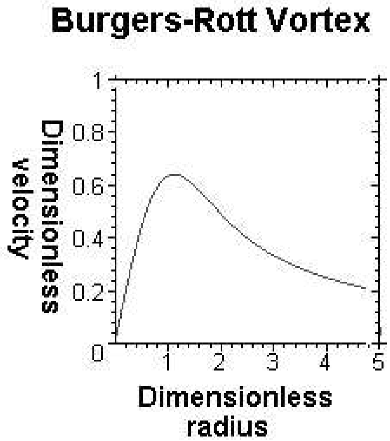

Figure 2: Form of azimuthal velocity in a Burgers-Rott vortex.

3.1 Stability of a Burgers-Rott vortex

By substituting the Burgers-Rott flow, V(r), in the Rayleigh stability criterion, one will notice that the entire vortex core is stable against radial disturbaces of flow because $r^2\Omega(r) = (1 - e^{-r^2})$ and the second derivative of the square of this quantity is > 0 for values of $r < 1.12$; and < 0 for larger values of r. (see the next page in this series on stability of radial disturbances). The tail flow may or may not be stable, but eddy viscosity should stablize it as long as the eddy viscosity is not a negative value.[9]

[7]Units of the abcissa are $(a/2\nu)^{\frac{1}{2}}$ and ordinate are $\frac{\Gamma(a/\nu)^{\frac{1}{2}}}{\pi 2^{\frac{3}{2}}}$.

[8]Molecular viscosity is 100,000 times smaller than typical eddy viscosity, and as I pointed out earlier, it makes no sense to apply it to any mechanical discussion herein.

[9]Refer once again to my earlier footnote about stability and negative viscosity.

4 The Sullivan vortex

The Sullivan vortex is also an exact solution to Navier-Stokes Equation. It has some similarity to the Burgers-Rott vortex. There is a one-celled vortex and a two-celled vortex. I am interested only in the two-celled vortex at this point.

The two-celled vortex has an inner cell in which air flow descends from above and flows outward to meet a separate air flow that is converging radially. Both flows rise at the point of meeting. The Sullivan vortex is probably the simplest vortex that can describe the flow in an intense tornado with a central downdraft, and it is the simplest vortex that localizes its updraft to a particular place–there is a place for the thunderstorm. The mathematical form of the Sullivan Vortex is:

$$\begin{aligned} U(r) &= -ar + \frac{6\nu}{r}(1 - e^{-\frac{ar^2}{2\nu}}) \\ V(r) &= \frac{\Gamma}{2\pi r}\frac{H(\frac{ar^2}{2\nu})}{H(\infty)} \\ W(z,r) &= 2az(1 - 3e^{-\frac{ar^2}{2\nu}}) \text{ where;} \\ P(r,z) &= P(r,z)_{Burgers-Rott} - \frac{18\rho\nu^2}{r^2}(1 - e^{-\frac{ar^2}{2\nu}})^2 \text{ where;} \end{aligned} \tag{5}$$

As in the Burgers-Rott vortex model, U,V, and W are the r, θ, and z components of velocity, respectively. Γ is circulation strength of the vortex, and a is a strength of suction. ν is a viscosity term, but it is eddy viscosity which dominates the value of this coefficient, not molecular viscosity. $H(x)$ is a function defined in terms of integrals; $H(x) = \int_0^x e^{f(t)}dt$ and in turn $f(t) = -t + 3\int_0^t (1 - e^{-y})\frac{dy}{y}$. Obviously the ratio $\frac{H(x)}{H(\infty)} \rightarrow 1$ as $x \rightarrow \infty$, and as a result the azimuthal velocity component eventually approaches that of the *potential flow* tail of the Rankine vortex.

$P(r,z)_{Burgers-Rott}$ is the distribution of pressure as in the Burgers-Rott vortex. And, as is plain to see, the difference between the two involves only a small correction factor. The axial pressure gradient is $\partial P/\partial z = -4\rho z a^2$, and increases vertically without bound. Once again this does not describe buoyancy well at all.

What is mathematically intriguing about the Sullivan vortex is that it is defined not in terms of elementary functions, but rather in terms of integrals of such. The integral for $f(t)$ in particular appears to diverge, which makes it interesting to calculate. Initially one might think that $H(x \to \infty)$ would diverge as it involves the integral $\int_0^\infty (1 - e^{-y}) \frac{dy}{y}$ which diverges toward *infinity*[10]. However, the divergence is pretty tame so that adding $-t$ sends it diverging toward *minus infinity*, and then exponentiating the result and integrating leads to near zero contribution to $H(x)$ beyond a value of about $x = 10$ or so. Thus beyond a value of argument to $H(x)$ of 10 or so the tail of the Sullivan vortex is essentially that of a Rankine combined vortex.[11] The figure below shows a normalized azimuthal velocity[12] being a smooth function of r. Note in particular that it has a definite slope at $r = 0$. Numerous authors over the years, including Sullivan, have drawn a figure indicating a flat slope at $r = 0$, and this even lead one researcher to suggest that azimuthal velocity is negligible within the inner cell. Both Mathematica and Maple have confirmed this linear limiting behavior of $H(x^2)/x$ as $x \to 0$, which one can also calculate easily by hand.

[10]Both Mathematica and Maple return a value of *Complex Infinity* for this integral.

[11]Note that the argument to H is $\frac{ar^2}{2\nu}$ which very rapidly grows to the point of $H(x) = Constant$. Also, and argument that is proportional to r^2 produces $\lim_{r \to 0} H(r^2)/r \to r$, and so the zero radius limit of the azimuthal velocity is linear in r.

[12]I have plotted the function $H(\rho^2)/\rho$ in this graph, where $\rho^2 = \frac{ar^2}{2\nu}$ because this illustrates the form of V_θ adequately without having to specify a and ν.

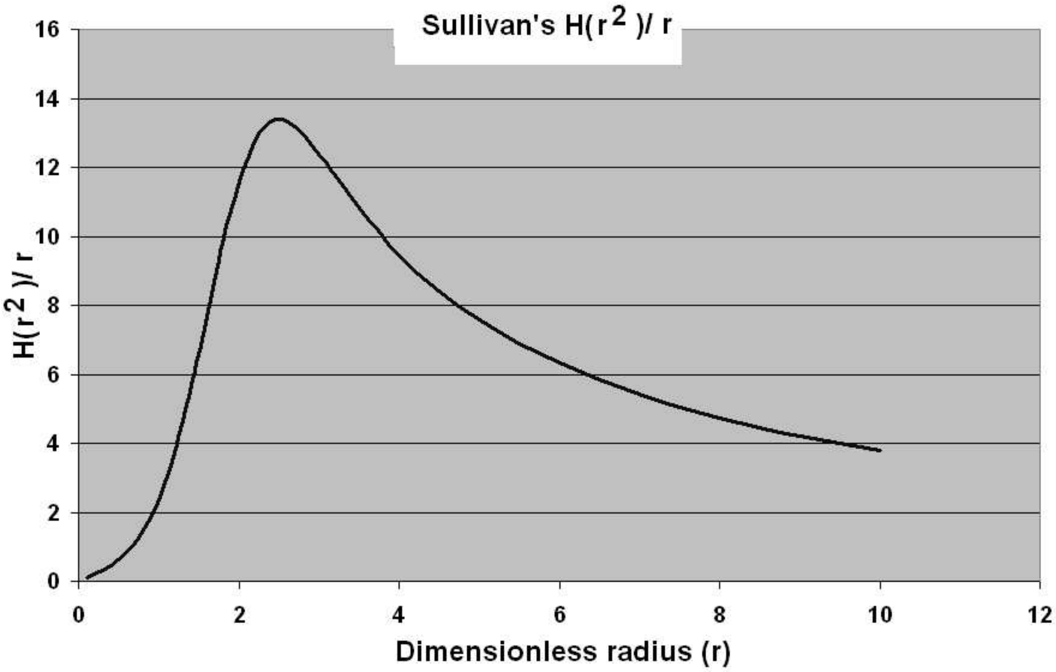

Figure 3: Form of azimuthal velocity in a two-celled Sullivan vortex. The plotted function is $H(x^2)/x$.

Similarly to the Burgers-Rott vortex, the Sullivan vortex places its suction at a great height, which I have pointed out will limit strict applicability to the near surface far below the LCL. It also places the vortex at the center of the updraft, and it is therefore too symmetric to describe the real tornado.

Davies-Jones[13] provides a thorough description comparison of both the Burgers-Rott and Sullivan vortexes. This is generally informative except for the statement that the Sullivan two-celled vortex may be unstable[14] because its *"vertical vorticity is concentrated in an annular region between the cells."* The accompanying figure shows the two-celled vortex has little azimuthal shear at the boundary between the cells, and is much like the Burger-Rott

[13]Thunderstorm Morphology and Dynamics, Edwin Kessler, Ed., Univ. of Oklahoma Press, 1981.

[14]More precisely *Barotropically unstable* but this is an unimportant point at this time.

and Rankine vortexes. It has vorticity concentrated in a core, and possesses a tail with nearly zero vorticity. Compare this to the nearby figure of velocity observed in a large *Texas-sized* dust devil.

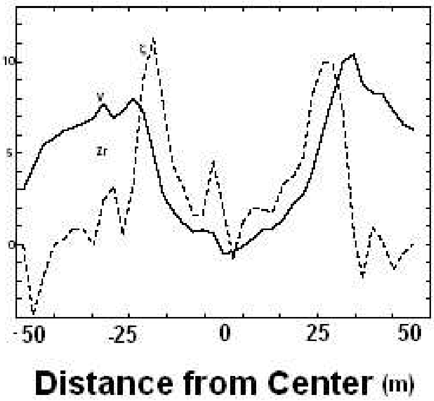

Figure 4: Radar derived velocity profile of a Texas-sized dustdevil. After measurements made and reported by Bluestein and Pazmany. The solid line is azimuthal velocity while the dashed is vertical component of vorticity.

If there is a region of concentrated vorticity within a Sullivan vortex, it is an azimuthally directed vorticity[15] resulting from the shear between downdraft and updraft within the inner cell. Using the definition of vorticity, $\vec{\omega} = \nabla \times \vec{V}$, I can show that the Sullivan vortex has two vorticity components:

The vertical component is ...[16]

[15]Closed threads of vorticity around the core of the vortex itself.

[16]γ is Euler's gamma here, not circulation.

$$\omega_z = \frac{\partial}{r\partial r}(rV_\theta(r))$$
$$= \frac{\Gamma a}{2\pi\nu H(\infty)}\partial H(r)/\partial r$$
$$\text{but because } H(r) = \int_0^r e^{f(t)}dt; \partial H(r)/\partial r = e^{f(r)}$$
$$\text{and so, } \omega_z = \frac{\Gamma a}{2\pi\nu H(\infty)}e^{f(r)}\text{with } f(r) = -r + 3(\gamma - Ei(-r)) \quad (6)$$

The azimuthal (θ) component is ...

$$\omega_\theta = (\frac{\partial V_r}{\partial z} - \frac{\partial V_z}{\partial r})$$
$$= -\frac{6a^2rz}{\nu}e^{-ar^2/2\nu}$$
$$(7)$$

The azimuthal component rises to a peak at $r_0 = (a/v)^{-\frac{1}{2}}$ and shows that the core wall is not a locus of great vertical vorticity, but rather one of great azimuthal vortcity. What is most interesting about this exercize is to combine the two components into a total vorticity vector $\vec{\omega_T} = \vec{\omega_z} + \vec{\omega_\theta}$ which, because $\omega_\theta < 0$ everywhere, represents helical vortex threads that wrap around the columnar tornado core with negative pitch–just like the suction vorticies often observed in potent tornadoes.

5 Kuo's vortexes

Kuo's work is particularly interesting in that it originated in finding what motions one could expect in a rotating air mass induced to convect. He found two solutions complete in the sense that he obtains both the mechanical quantities (velocities for example) and also the thermodynamic quantities. One is single-celled and the other is two-celled. The one-celled vortex has no closed form solution and has to be calculated approximately by numerical means. The two-celled vortex has a closed form if one can assume that viscous and thermal eddy diffusion coefficients are equal (i.e. an eddy Prandtl number of 1). It is ...

$$
\begin{aligned}
U(R) &= -bR(\frac{1}{2} - (1 - e^{-\frac{R^2}{4\nu}})\frac{4\nu}{R^2})) \\
V(R) &= \frac{\Gamma m_0}{\sqrt{2}R} \\
W(z) &= bz(1 - 2e^{-\frac{R^2}{4\nu}})
\end{aligned}
\tag{8}
$$

where, b is the square root of a stability factor, and plays the role that the strength of suction (a) does in the Burgers-Rott and Sullivan vortexes. This factor has a value of about 0.01 s^{-1} for moderate instability, ν is eddy viscosity with a typical value of 5 m^2/s. R is a dimensionless radial coordinate equal to r/h, where h is a scale height. $m_0 = \int_0^x e^{f(t)}dt / \int_0^\infty e^{f(t)}dt$[17], $f(x) = (2 - x)\int_0^x (1 - e^{-t})\frac{dt}{t}$; and Γ is the circulation of the vortex, which is of the order of 8000 m^2/s in a large tornado. The reader will, no doubt, notice the similarity of the Sullivan vortex and Kuo's vortex. Both the vertical motion and inflow depend on the stability factor, while the azimuthal velocity of the vortex depends on the circulation (Γ) but not on the stability factor.

Possibly there are other numerical steady-state solutions of the Navier-Stokes system, but I doubt there is much point in pursuing them. In the next section I'll suggest serious shortcomings of steady state solutions.

6 Pertinent observations

All three of the real vortex models above, the models that are solutions of the Navier-Stokes equation, are steady state. What this means is that they represent solutions of the Navier-Stokes equation in which any term involving $\partial/\partial t$ is gone, and generally this implies a solution that one obtains in the

[17]Note the similarity to Sullivan's $H(x^2)/H(\infty)$.

limit that $t \to \infty$. With viscosity included, and the solutions possessing a boundary layer, which the Burgers-Rott, Sullivan, and Kuo vortexes do include, the Navier-Stokes partial differential equation will exhibit a diffusive behavior (*parabolic PDE form*). Thus, the situation of $t \to \infty$, results in the complete obliteration of any information regarding initial conditions that lead to the tornado initiation and evolution. I mentioned that all three solutions apear similar to one another. The parabolic flavor of the Navier-Stokes equation combined with $t \to \infty$ almost demands this. There is, in effect no information left regarding the initial mechanical state of the atmosphere except its circulation (Γ), which is an arbitrary value left to the discretion of the modeller; and, what seems more pertinent to me, there is no reason to favor one of these models over another. Kuo fit observed data to his model and finding a reasonable agreement between the two, pronounced his model essentially correct. Yet the non-real Rankine combined vortex fits equally well. I am more inclined to think that the fit between data from real tornadoes and these models suggests not that the models are correct, but instead that tornadoes are features in which initial conditions are obliterated, diffused or convected away, and that only a knowledge of Γ remains.

Some other speculations that I might tackle in the future:

- Since the flow of these models is such that $\partial/\partial t = 0$, $t \to \infty$; how long must the flow persist in order to reasonably approximate $t \to \infty$? We have no model solutions involving time evolution to answer this question.

- Even if we cannot find model solutions with time dependence, can we use the complete Navier-Stokes equation to argue for explosive growth of a vortex versus just exponential growth? Explosive growth would mean that the vortex achieves a steady-state form in a finite time. It seems obvious to me that real tornadoes behave so.

- Why is the initial value of Γ preserved? Why isn't it diffused and convected away by the flow as $t \to \infty$? Surely $t \to \infty$ is plenty of time for diffusion, no matter how small is the value of ν, to destroy all initial information including Γ. We will never understand this mystery with the present models because Γ is arbitrary and the azimuthal flow is effectively uncoupled from the radial flow which is alleged to concentrate Γ.

- Might Γ be found as a function of the flow itself, rather than an arbitrary number set *a priori*?
- Can we find model equations in which the tornado is a *bootstrapping* feature? That is to say, a feature that generates itself by devouring vorticity produced in the boundary layer of its own inflow?

I plan to investigate some of these issues, and in successive installments of this story, I'll report what I have learned.