Euler's disc and its finite-time singularity

By
Kevin T. Kilty,
April, 2000

Note to readers

This contribution I sent to *Nature* in response to a Letter to Nature, 20 April, 2000, p.855, by Dr. Keith Moffatt, who was gracious enough to correspond with me on the topic. Nature's reviewer stated that it was "beneath the dignity" of Nature to consider publishing such trivia as I had submitted. Nature published the "trivia" that prompted it, however. Moffat has proposed that air friction brings the Euler disc to a halt in a time that is independent of the surface on which it spins. My intention was to show that a dependence on surface properties results in the same dynamic behavior, and is more in line with common experience. What follows is my original response, verbatim.

Dear Editor,

The correspondence regarding Euler's disc (April 20) was very interesting. However, over the years I have taken countless coins from my pocket and tossed them onto sales counters, thrown them onto furniture, or dropped them on floors. All of this experience has proved two things. First, that a coin shudders to a stop within a few seconds even when it has begun its spin completely upright. The equivalent time based on the theory of your correspondence would be orders of magnitude longer. Second, a coin shudders to a halt more quickly on a rough, soft surface, such as varnished fir, than it does on a smooth hard surface such as a glass sales counter.

I propose a modification of the theory. Let me refer to the analysis in your brief communication and diverge from it only after the first two equations.

$$\Omega^2 \alpha = 4\frac{g}{a} \tag{1}$$

Equation 1 relates the rate of spin to the angle that the disc makes with the surface (α), the acceleration of gravity (g) and the disc radius (a). Equation 2,

$$E = \frac{3}{2} Mga\, sin(\alpha) \tag{2}$$

where (M) is the mass of the disc, represents total mechanical energy.

Consider rolling friction rather than air viscosity as the source of dissipation. When an elastic disc rolls on an elastic surface both the disc and the surface deform. This produces a small component of force that resists the rolling motion. Typically we write this force as being proportional to the weight of the disc, which is $f = \mu Mg$, where μ is called the coefficient of rolling friction, as long as the acceleration of α is small.[1] We call this rolling friction even though it has little to do with friction in the usual sense. If we multiply this force by the speed of the point of contact between the disc and surface, Ωa, the result is a rate of dissipation equal to $-\mu\Omega Mga$. Therefore, Equation 3 becomes[2]

$$E = \frac{3}{2} Mga \frac{d\alpha}{dt} = -\mu\Omega Mga \tag{3}$$

which integrates to

$$\alpha^{\frac{3}{2}}(t) = \frac{t - t_0}{t_1} \tag{4}$$

where, $t_1 = \frac{(a/g)^{\frac{1}{2}}}{2\mu}$.

This equation displays a finite-time singularity. The disc will settle to the surface in a time $t_0 = \alpha_0^{\frac{3}{2}} t_1$, which is the correct order of magnitude for a coin begun at $\alpha_0 = 0.5$ or greater as long as μ is about 0.001. Gravity acting on the mass of the disc provides only limited torque, which limits the acceleration of α to a value less than $\frac{4g}{5a}$. This has the further interesting consequence that as the coin settles, the vertical reaction at the point of contact diminishes and so does the rolling friction. In other words, before the coin reaches its limiting acceleration viscous dissipation has truly become the dominant effect, and the coin is now so nearly flat to the surface that an escaping cushion of air limits its fall. Thus, the coin follows an evolving dynamic that begins with rolling friction, passes through a phase of viscous

[1] This may be a confusing usage of term. What I mean is that the second derivative with respect to time of α should be small so that we can consider the weight of the disk as constant.

[2] Equation 3 in Moffat's original letter to Nature.

braking, and ends with a cushioned fall to the surface.
Very truly yours,

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Theory left out of the letter to Nature

Assume that the Euler Disc has the problem geometry of Figure 1.

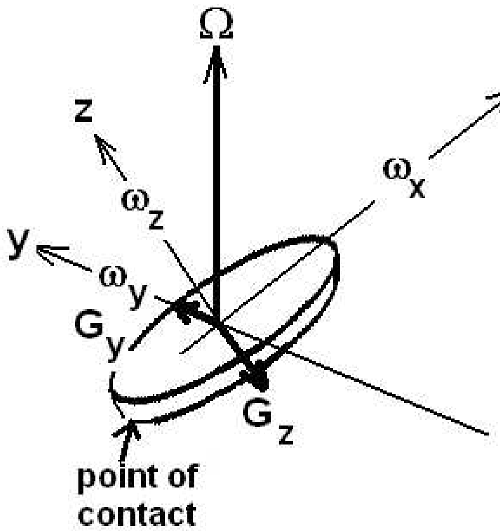

Figure 1: Explanation of the problem geometry. The vector Ω represents the precession of the disc, while ω represents angular velocity of rotation along the three geometric axes. G_y and G_z are torques supplied by the weight of the disc. The angle between Ω and $\hat{z}$ is α which is obviously a function of time.

Euler's equations for the problems are:

$$
\begin{aligned}
I_{xx}\dot{\omega_x} - (I_{yy} - I_{zz})\omega_y\omega_z &= G_x \\
I_{yy}\dot{\omega_y} - (I_{zz} - I_{xx})\omega_x\omega_z &= G_y \\
I_{zz}\dot{\omega_z} - (I_{xx} - I_{yy})\omega_x\omega_y &= G_z
\end{aligned} \tag{5}
$$

It is apparent that $\omega_x = \Omega\, Cos(\alpha)$ so that $\dot{\omega_x} = \dot{\Omega} Cos(\alpha) - \Omega\, Sin(\alpha)\dot{\alpha}$; ω_y stays so small throughout the motion that we can consider it as 0; and, $\omega_z = \Omega\, Sin(\alpha)$ so that $\dot{\omega_z} = \dot{\Omega} Sin(\alpha) + \Omega\, Cos(\alpha)\dot{\alpha}$. The torque that gravity supplies is $G_x = 0$, $G_y = Mga\, Cos(\alpha)$, and $G_z = -N$, where N is a normal reaction at the point of contact. N will vary as the motion evolves because of the acceleration of α. Assuming that α is always a small quantity (true except early in the motion), so that the typical linear approximations to the trig functions are available, provides the following approximate system of Euler equations ...

$$
\begin{gathered}
I_{xx}\dot{\omega_x} = 0 \\
I_{yy}\dot{\omega_y} - \frac{Ma^2}{4}\omega_x\omega_z = Mga \\
I_{zz}\dot{\omega_z} = -N \\
\text{or} \\
\dot{\Omega} = \frac{4g}{a} - \Omega^2\alpha \\
\dot{\Omega}\alpha + \Omega\dot{\alpha} = \frac{-2N}{Ma}
\end{gathered} \tag{6}
$$

Ingoring $\dot{\Omega}$, which is small, in these equations results in the system of equations that Moffat used for his analysis.

$$
\begin{aligned}
\frac{4g}{a} &= \Omega^2 \alpha \\
\Omega \dot{\alpha} &= \frac{-2N}{Ma}
\end{aligned}
\tag{7}
$$

The problem now becomes that of figuring how N varies with α so that we can integrate these equations of motion. Because most mathematics students now have access to one or another of the computer algebra systems[3] it ought to be possible to integrate a much less approximate system equations. More about this after the next school year.

Some experimental data

A few months later I decided to perform an experiment on a Sacajawea Dollar as an Euler Disc to determine whether the coin would fall to the surface according to α^3 proportional to time, which is what I expect of viscous braking, or according to $\alpha^{\frac{3}{2}}$ proportional to time which is what I expect of rolling friction.

[3]Such as Maple or Mathematica.

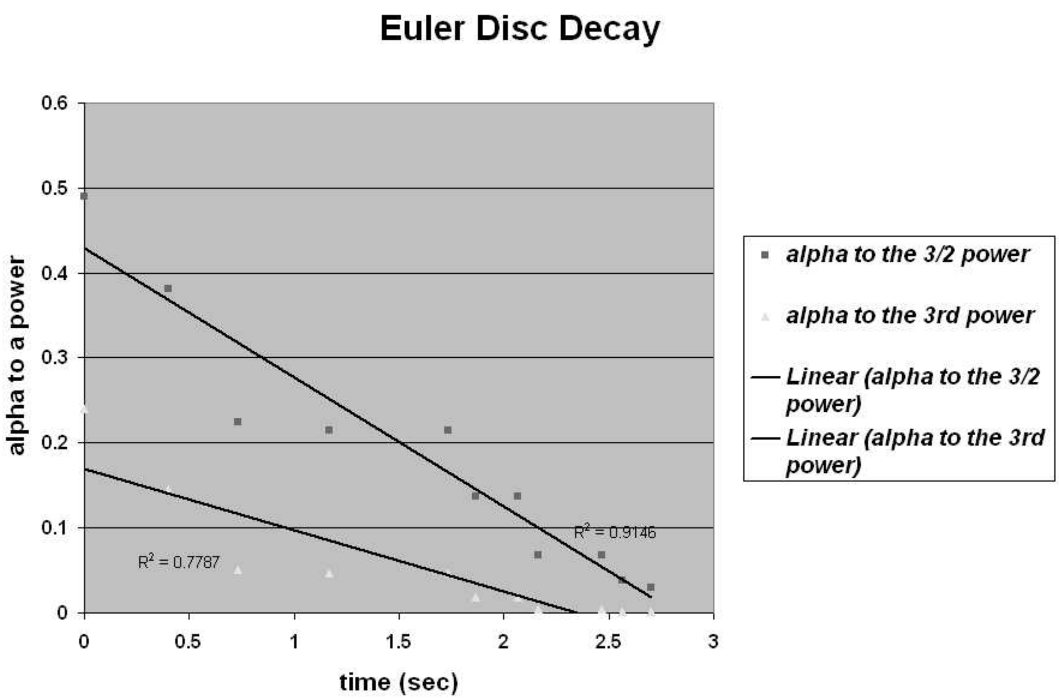

Figure 2: Time decay of an Euler Disc. The angle, α, is raised to the $\frac{3}{2}$, and $3rd$ powers respectively. Trendlines show that $\frac{3}{2}$ makes a more linear fit to increasing time, but only marginally so.

Apparently $\frac{3}{2}$ makes a better fit of a straight line to linearly increasing time, but it is only a marginally better fit, and it makes the best fit at late-time in the motion, which is contrary my statement in the letter to Nature that by late-time the normal force, N, has diminished. This is not so, $\frac{d^2\alpha}{dt^2}$, is nearly zero at late time so the normal force is nearly the coin's weight, just as it is early in the motion.